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SHORT COMMUNICATION

PROPAGATION OF RAYLEIGH WAVES IN A HETEROGENEOUS INCOMPRESSIBLE SUBSTRATUM OVER A HOMOGENEOUS INCOMPRESSIBLE HALF-SPACE

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SUMMARY

The paper studies the propagation of Rayleigh waves in an incompressible layer with general variation of rigidity as $\mu = \mu_0(1 + bz)^m$ resting over a homogeneous incompressible elastic half-space. Instead of using the Whittaker function, the expansion formula proposed by Newlands[3] has been used for better results at shallow depths. As a particular case for $m = 1$, the results have been shown to coincide with those obtained by Newlands.[3] The velocities have been computed for different values of m and the results are presented in graphs.

KEY WORDS: Rayleigh waves; propagation; heterogeneous incompressible substratum; homogeneous incompressible half-space

INTRODUCTION

The study of propagation of Rayleigh waves in the earth's crust must include the changing properties of the crust. The earth's crust is stratified and so a study of this nature requires consideration of at least a few layers over a half-space. Further, these layers are heterogeneous in density and rigidity.

A good amount of information about the propagation of Rayleigh waves in a superficial layer over a half-space with different variation in density and rigidity is available in Ewing *et al.*[1] In some of the studies reported therein, the solutions are expressed in terms of Whittaker functions[2] and the computation involves asymptotic expansions. However, the asymptotic expansions are only valid at large depths but do not give good results at shallow depths. Newlands[3] offered a new approach to the problem and obtained a solution which may be used at shallow depths. He derived the velocity of propagation of Rayleigh waves in a layer with variation in rigidity $\mu = \mu_2 + \mu_1 z$, where μ_1 and μ_2 are constants. Recently, Vrettos[4,5] has studied the propagation of Rayleigh waves in a half-space due to a line load with shear modulus variation $\mu = \mu_0 + (\mu_\infty - \mu_0)[1 - \exp(-az)]$, $0 < \mu_0 \leqslant \mu_\infty$, using a series solution of the governing equations.

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Following Newlands,[3] the propagation of Rayleigh waves in an incompressible heterogeneous medium of shear modulus $\mu = \mu_0(1 + bz)^m$ is studied in this paper for various values of m. The results of this paper coincide with those of Newlands[3] for the particular case of $m = 1$.

FORMULATION

Consider an incompressible layer of thickness H with shear modulus $\mu = \mu_0(1 + bz)^m$ and density $\rho = \rho_0$ over a half-space with constant shear modulus μ_2 and density ρ_2, z being the vertical distance from the origin, located at the interface of the layer and the half-space. The downward direction has been taken as positive for z.

Consider a harmonic plane wave propagating along the x direction with wave velocity C and wavelength $2\pi/K$. Let u, w be the displacement components in the x, z directions, respectively, at a point (x, y, z) at any time and suppose that apart from a factor $e^{iK(x-Ct)}$, u and w are functions of z only.

GOVERNING EQUATIONS AND SOLUTION FOR THE LAYER

The equation of vibratory motion in two dimensions for an elastic solid are

$$\frac{\partial}{\partial x}\left\{\lambda\Delta + 2\mu\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} = \rho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{\lambda\Delta + 2\mu\frac{\partial w}{\partial z}\right\} = \rho\frac{\partial^2 w}{\partial t^2} \tag{2}$$

where λ is Lame's constant, Δ is dilatation, μ is shear modulus and ρ is the denisty of the medium.

For an incompressible medium

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{3}$$

Using the relations

$$\begin{aligned} u &= \frac{\partial \phi}{\partial z} + \frac{\partial \chi}{\partial x} \\ w &= \frac{\partial \phi}{\partial z} - \frac{\partial \chi}{\partial x} \end{aligned} \tag{4}$$

ϕ and χ being scalar and vector potentials respectively. Using $\mu = \mu_0(1 + bz)^m$ as well as the incompressibility condition, $\Delta = 0$, together with $\lim_{\lambda\to\infty,\Delta\to 0}\lambda\Delta \to -P_1$. P_1 being the hydrostatic stress, equations (1) and (2) take the form

$$\frac{\partial}{\partial x}[-P_1 + 2\mu_0 bm(1 + bz)^{m-1}w + \rho K^2C^2\phi] + \frac{\partial}{\partial z}[\mu\nabla^2\chi + \rho K^2C^2\chi] = 0 \tag{5}$$

$$\frac{\partial}{\partial z}[-P_1 + 2\mu_0 bm(1 + bz)^{m-1}w + \rho K^2C^2\phi] - \frac{\partial}{\partial x}[\mu\nabla^2\chi + \rho K^2C^2\chi] = 0 \tag{6}$$

which are satisfied by

$$P_1 = 2\mu_0 bm(1 + bz)^{m-1} w + \rho K^2 C^2 \phi \tag{7}$$

$$\mu \nabla^2 \chi = - \rho K^2 C^2 \chi \tag{8}$$

together with

$$\nabla^2 \phi = 0 \tag{9}$$

Equations (8) and (9) are equivalent to

$$d^2\chi/dz^2 + K^2\left[\frac{\rho^2 C^2}{\mu_0(1 + bz)^m} - 1\right]\chi = 0 \tag{10}$$

$$d^2\phi/dz^2 - K^2\phi = 0 \tag{11}$$

Now taking $(1 + bz) = Z$, equation (10) takes the form

$$\frac{d^2\chi}{dz^2} - \left(\frac{K}{b}\right)^2\left(1 - \frac{\rho C^2}{\mu_0 Z^m}\right)\chi = 0 \tag{12}$$

The series solution of equation (12) in powers of $(K/b)^2$ may be written as

$$\chi(Z) = \chi_0(Z) + (K/b)^2 \chi_1(Z) + \cdots + (K/b)^{2n}\chi_n(Z) + \cdots \tag{13}$$

so that

$$\left[\chi_0'' + \left(\frac{K}{b}\right)^2 \chi_1'' + \cdots\right] - \left(\frac{K}{b}\right)^2\left(1 - \frac{\rho C^2}{\mu_0 Z^m}\right)[\chi_0 + (K/b)^2\chi_1 + \cdots] = 0 \tag{14}$$

giving

$$\chi_0'' = 0, \quad \chi_1'' = \left(1 - \frac{\rho C^2}{\mu_0 Z^m}\right)\chi_0, \ldots, \chi_{n+1}'' = \left(1 + \frac{\rho C^2}{\mu_0 Z^m}\right)\chi_n \tag{15}$$

For equation (13) to converge we must have

$$\chi_n'(1) = \chi_n(1) = 0 \tag{16}$$

Further, from equation (15) one may get

$$\chi_0'(Z) = \frac{A_2}{b} \tag{17}$$

$$\chi_0(Z) = A_1 + \frac{A_2}{b}(Z - 1) \tag{18}$$

From the linearity of (12) one may write

$$\begin{aligned}\chi &= A_1\left\{\chi_0^{(1)} + \left(\frac{K}{b}\right)^2 \chi_1^{(1)} + \cdots \quad \cdots \quad \cdots\right\} \\ &\quad + A_2\left\{\chi_0^{(2)} + \left(\frac{K}{b}\right)^2 \chi_1^{(2)} + \cdots \quad \cdots \quad \cdots\right\} \\ &= A_1\chi^{(1)} + A_2\chi^{(2)}\end{aligned} \tag{19}$$

where

$$\chi_0^{(1)} = 1 \tag{20}$$

$$\chi_0^{(2)} = z = \left(\frac{Z-1}{b}\right)$$

From equation (15), χ_1 may be obtained as

$$\chi_1 = \int_1^Z \mathrm{d}\xi \int_1^\xi \left(1 - \frac{\rho C^2}{\mu_0 t^m}\right)\chi_0(t)\,\mathrm{d}t$$

This integral can be evaluated for different values of m as follows:

Case I: When $m = 1$

$$\chi_1 = A_1\left[\frac{1}{2}(Z-1)^2 - \frac{\rho C^2}{\mu_0}\{Z\log Z - (Z-1)\right] + \frac{A_2}{b}\left[\frac{1}{6}(Z-1)^3 + \frac{\rho C^2}{\mu_0}\left(\frac{1}{2} - \frac{Z^2}{2} + Z\log Z\right)\right]$$

The solutions of $\chi^{(1)}$ and $\chi^{(2)}$ are

$$\chi_{(Z)}^{(1)} = 1 + \left(\frac{K}{b}\right)^2\left[\frac{1}{2}(Z-1)^2 - \frac{\rho C^2}{\mu_0}\{Z\log Z - (Z-1)\}\right] \tag{21}$$

$$\chi_{(Z)}^{(2)} = (Z-1) + \left(\frac{K}{b}\right)^2\left[\frac{1}{6}(Z-1)^3 + \frac{\rho C^2}{\mu_0}\left(\frac{1}{2} - \frac{Z^2}{2} + Z\log Z\right)\right] \tag{22}$$

Case II: When $m = 2$

$$\chi_1 = A_1\left[\frac{(Z-1)^2}{2} + \frac{\rho C^2}{\mu_0}\{\log Z - (Z-1)\}\right] + \frac{A_2}{b}\left[\frac{1}{6}(Z-1)^3 - \frac{\rho C^2}{\mu_0}\{(Z+1)\log Z + 2(1-Z)\}\right]$$

and the solutions of $\chi^{(1)}$ and $\chi^{(2)}$ are

$$\chi_{(Z)}^{(1)} = 1 + \left(\frac{K}{b}\right)^2\left[\frac{(Z-1)^2}{2} + \frac{\rho C^2}{\mu_0}\{\log Z - (Z-1)\}\right] \tag{23}$$

$$\chi_{(Z)}^{(2)} = (Z-1) + \left(\frac{K}{b}\right)^2\left[\frac{1}{6}(Z-1)^3 - \frac{\rho C^2}{\mu_0}\{(Z+1)\log Z + 2(1-Z)\}\right] \tag{24}$$

Case III: When $m = 3$

$$\chi_1 = A_1\left[\frac{(Z-1)^2}{2} + \frac{\rho C^2}{\mu_0}\left(1 - \frac{Z}{2} - \frac{1}{2Z}\right)\right] + \frac{A_2}{b}\left[\frac{(Z-1)^3}{6} + \frac{\rho C^2}{\mu_0}\left(\log Z + \frac{1}{2Z} - \frac{Z}{2}\right)\right]$$

and the solutions of $\chi^{(1)}$ and $\chi^{(2)}$ are

$$\chi_{(Z)}^{(1)} = 1 + \left(\frac{K}{b}\right)^2\left[\frac{(Z-1)^2}{2} + \frac{\rho C^2}{\mu_0}\left(1 - \frac{Z}{2} - \frac{1}{2Z}\right)\right] \tag{25}$$

$$\chi_{(Z)}^{(2)} = (Z-1) + \left(\frac{K}{b}\right)^2\left[\frac{1}{6}(Z-1)^3 + \frac{\rho C^2}{\mu_0}(\log Z + \frac{1}{2Z} - \frac{Z^2}{2})\right] \tag{26}$$

Case IV: For any value of m but $m \neq 1$, $m \neq 2$, $m \neq 3$,

$$\chi_1 = A_1\left[\frac{1}{2}(Z-1)^2 + \frac{\rho C^2}{\mu_0}\frac{1}{(1-m)}\left\{(Z-1) + \frac{1}{(2-m)}\left(1 - \frac{1}{Z^{m-2}}\right)\right\}\right]$$

$$+ \frac{A_2}{b}\left[\frac{1}{6}(Z-1)^3 + \frac{\rho C^2}{\mu_0}\frac{1}{(2-m)}\left\{(Z-1) + \frac{1}{(3-m)}\left(1 - \frac{1}{Z^{m-3}}\right)\right\}\right.$$

$$\left. + \frac{\rho C^2}{\mu_0}\frac{1}{(1-m)}\left\{(1-Z) + \frac{1}{(2-m)}\left(\frac{1}{Z^{m-2}} - 1\right)\right\}\right]$$

and the solutions of $\chi^{(1)}$ and $\chi^{(2)}$ are

$$\chi^{(1)}(Z) = 1 + \left(\frac{K}{b}\right)^2\left[\frac{(Z-1)^2}{2} + \frac{\rho C^2}{\mu_0}\frac{1}{(1-m)}\left\{(Z-1) + \frac{1}{(2-m)}\left(1 - \frac{1}{Z^{m-2}}\right)\right\}\right] \tag{27}$$

$$\chi^{(2)}(Z) = (Z-1) + \left(\frac{K}{b}\right)^2\left[\frac{(Z-1)^3}{6} + \frac{\rho C^2}{\mu_0(2-m)}\left\{(Z-1) + \frac{1}{(3-m)}\left(1 - \frac{1}{Z^{m-3}}\right)\right\}\right.$$

$$\left. + \frac{\rho C^2}{\mu_0(1-m)}\left\{(1-Z) + \frac{1}{(2-m)}\left(\frac{1}{Z^{m-2}} - 1\right)\right\}\right] \tag{28}$$

Hence the solutions are

$$\begin{aligned} \phi &= (P\cosh Kz + Q\sinh Kz)\cos K(x - Ct) \\ \chi &= (A_1\chi^{(1)}(z) + A_2\chi^{(2)}(z))\sin K(x - Ct) \end{aligned} \tag{29}$$

where $\chi^{(1)}(z)$ and $\chi^{(1)}(z)$ have the values given by equations (21)–(28), depending upon the values of m.

The displacement and stress components in the layer have been calculated as

$$u = \left[-K(P\cosh Kz + Q\sinh Kz) + \left(A_1\frac{\partial}{\partial z}\chi^{(1)}(Z) + A_2\frac{\partial}{\partial z}\chi^{(2)}(Z)\right)\right]\sin K(x - Ct)$$

$$w = [K(P\sinh Kz + Q\cosh Kz)\,K(A_1\chi^{(1)}(Z) + A_2\chi^{(2)}(Z)]\cos K(x - Ct)$$

$$\sigma_{xz} = 2K\left[-\mu\frac{\partial\phi}{\partial z} + K\left(\mu - \frac{\varepsilon\mu_0}{2}\right)\chi\right]$$

$$\sigma_{zz} = 2\left[K^2\phi\left(\mu - \frac{\varepsilon\mu_0}{2}\right) - \mu_0 bm(1 + bz)^{m-1}\frac{\partial\phi}{\partial z} + \mu_0 bm(1 + bz)^{m-1}K\chi - \mu K\frac{\partial\chi}{\partial z}\right]$$

where $\varepsilon = C^2/(\mu_0/\rho)$.

Solution for the homogeneous half-space

The values of ϕ and χ may be obtained as

$$\begin{aligned} \phi &= R\,e^{-Kz}\cos K(x - Ct) \\ \chi &= S\,e^{-Knz}\sin K(x - Ct) \end{aligned} \tag{30}$$

where

$$n^2 = \left(1 - \frac{\rho C^2}{\mu_2}\right).$$

The displacement and stress components in the half-space obtained as

$$u = (-KR\,\mathrm{e}^{-Kz} - SKn\,\mathrm{e}^{-Knz})\sin K(x - Ct)$$

$$w = (-RK\,\mathrm{e}^{-Kz} - SK\,\mathrm{e}^{-Knz})\cos K(x - Ct)$$

$$\sigma_{xz} = 2K\left[-\mu\frac{\partial\phi}{\partial z} + K(\mu - \tfrac{1}{2}\varepsilon\delta\mu_0)\chi\right]$$

$$\sigma_{zz} = 2\left[K^2\phi(\mu - \tfrac{1}{2}\varepsilon\mu_0\delta) - \mu K\frac{\partial\chi}{\partial z}\right]$$

where $\delta = \rho_2/\rho_0$.

Boundary conditions

The boundary conditions are (see Figure 1)

(i) u, w, σ_{xz}, σ_{zz} are continuous at $z = 0$,
(ii) σ_{xz} and σ_{zz} vanish at $z = -H$.

Set (i) gives

$$\begin{gathered}
-KP + A_2 b = -KR - KnS \\
Q - A_1 = -R - S \\
-Q + \left(1 - \frac{\varepsilon}{2}\right)A_1 = \frac{\mu_2 R}{\mu_0} + \left(\frac{\mu_2}{\mu_0} - \frac{\varepsilon\delta}{2}\right)S \\
KP\left(1 - \frac{\varepsilon}{2}\right) - bmQ + bmA_1 - bA_2 = KR\left(\frac{\mu_2}{\mu_0} - \frac{\varepsilon\delta}{2}\right) + \frac{\mu_2}{\mu_1}KnS
\end{gathered} \tag{31}$$

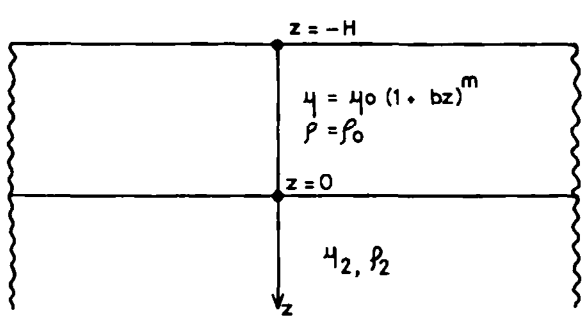

Figure 1. Model of the problem

and set (ii) gives

$$P\cosh(-KH) + Q\cosh(-KH) - \left(1 - \frac{\varepsilon\mu_0}{2\mu_1}\right)[A_1\chi^{(1)}(z) + A_2\chi^{(2)}(z)]_{z=-H} = 0$$

$$\begin{aligned} &P[K(\mu_1 - \tfrac{1}{2}\varepsilon\mu_0)\cosh(-KH) - \mu_0 bm(1+bz)^{m-1}\sinh(-KH)]_{z=-H} \\ &\quad + Q[K(\mu_1 - \tfrac{1}{2}\varepsilon\mu_0)\sinh(-KH) - \mu_0 bm(1+bz)^{m-1}\cosh(-\mathrm{KH})]_{z=-H} \\ &\quad + A_1\left[\mu_0 bm(1+bz)^{m-1}\chi^{(1)}(z) - \mu_1\frac{\partial}{\partial z}\chi^{(1)}(z)\right]_{z=-H} \\ &\quad + A_2\left[\mu_0 bm(1+bz)^{m-1}\chi^{(2)}(z) - \mu_1\frac{\partial}{\partial z}\chi^{(2)}(z)\right]_{z=-H} = 0 \end{aligned} \tag{32}$$

where $\mu_1 = \mu_0(1 - bH)^m$, the shear modulus at the surface.

From equation (31) the relationship between P, Q, A_1, A_2 may be obtained as

$$\begin{aligned} \tfrac{1}{2}\varepsilon P &= a_1 R + b_1 S \\ \tfrac{1}{2}\varepsilon Q &= a_2 R + b_2 S \\ \tfrac{1}{2}\varepsilon A_1 &= a_3 R + b_3 S \\ \tfrac{1}{2}\varepsilon A_2 &= a_4 R + b_4 S \end{aligned} \tag{33}$$

where

$$a_1 = \frac{1}{K}\left[K - \frac{bma_2}{\varepsilon/2} + \frac{bma_3}{\varepsilon/2} - K\left(\frac{\mu_2}{\mu_0} - \frac{\varepsilon\delta}{2}\right)\right]$$

$$a_2 = a_3 - \frac{\varepsilon}{2}$$

$$a_3 = 1 - \frac{\mu_2}{\mu_0}$$

$$a_4 = \frac{1}{b}\left(Ka_1 - \frac{\varepsilon K}{2}\right)$$

$$b_1 = \frac{1}{K}\left[Kn - \frac{bmb_2}{\varepsilon/2} + \frac{bmb_3}{\varepsilon/2} - \frac{\mu_2}{\mu_0}Kn\right)\Bigg]$$

$$b_2 = b_3 - \frac{\varepsilon}{2}$$

$$b_3 = \frac{1}{2} - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2}$$

$$b_4 = \frac{1}{b}\left(Kb_1 - \frac{\varepsilon Kn}{2}\right)$$

Now writing

$$C_1 = \sinh(-KH)$$
$$C_2 = \cosh(-KH)$$
$$C_3 = \left[-\left(1 - \frac{\varepsilon\mu_0}{2\mu_1}\right)\chi^{(1)}(z)\right]_{z=-H} \tag{34}$$
$$C_4 = \left[-\left(1 - \frac{\varepsilon\mu_0}{2\mu_1}\right)\chi^{(2)}(z)\right]_{z=-H}$$
$$d_1 = \left[K\left(\mu_1 - \frac{\varepsilon\mu_0}{2}\right)\cosh(-KH) - \mu_0 bm(1+bz)^{m-1}\sinh(-KH\right]_{z=-H}$$
$$d_2 = \left[K\left(\mu_1 - \frac{\varepsilon\mu_0}{2}\right)\sinh(-KH) - \mu_0 bm(1+bz)^{m-1}\cosh(-KH\right]_{z=-H}$$
$$d_3 = \left[\mu_0 bm(1+bz)^{m-1}\chi^{(1)}(z) - \mu_1\frac{\partial}{\partial z}\chi^{(1)}(z)\right]_{z=-H} \tag{35}$$
$$d_4 = \left[\mu_0 bm(1+bz)^{m-1}\chi^{(2)}(z) - \mu_1\frac{\partial}{\partial z}\chi^{(2)}(z)\right]_{z=-H}$$

The consistency of equation (33) for a non-trivial solution of R and S implies that

$$\sum_{i=1}^{4} a_i C_i \sum_{j=1}^{4} b_j d_j = \sum_{k=1}^{4} a_k d_k \sum_{l=1}^{4} b_l C_l \tag{36}$$

This is the velocity equation for the Rayleigh waves propagating in the medium under discussion.

Particular cases

Case I: When the upper layer is homogeneous (i.e. $m = 0$), then

$$a_1 C_1 = \left[1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2}\right]\sinh(-KH)$$
$$a_2 C_2 = \left[1 - \frac{\mu_2}{\mu_0} - \frac{\varepsilon}{2}\right]\cosh(-KH)$$
$$a_3 C_3 = \left(1 - \frac{\mu_2}{\mu_0}\right)\left(\frac{\varepsilon\mu_0}{2\mu_1} - 1\right)\left[1 + \frac{K^2H^2}{2}\left(1 - \frac{\rho C^2}{\mu_0}\right)\right]$$
$$a_4 C_4 = \frac{KH}{6}\left[1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2} - \frac{\varepsilon}{2}\right]\left(\frac{\varepsilon\mu_0}{2\mu_1} - 1\right)\left[-1 + K^2H^2\left(\frac{\rho C^2}{\mu_0} - 1\right)\right]$$
$$b_1 d_1 = Kn\left(1 - \frac{\mu_2}{\mu_0}\right)\left(\mu_1 - \frac{\varepsilon\mu_0}{2}\right)\cosh(-KH)$$
$$b_2 d_2 = K\left(1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2} - \frac{\varepsilon}{2}\right)\left(\mu_2 - \frac{\mu_0}{2}\right)\sinh(-KH)$$
$$b_3 d_3 = \mu_1 K^2 H\left(1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2}\right)\left(1 - \frac{\rho C^2}{\mu_0}\right)$$
$$b_4 d_4 = 0$$

$$a_1d_1 = K\left(1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2}\right)\left(\mu_1 - \frac{\varepsilon\mu_0}{2}\right)\cosh(-KH)$$

$$a_2d_2 = K\left(1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon}{2}\right)\left(\mu_1 - \frac{\varepsilon\mu_0}{2}\right)\sinh(-KH)$$

$$a_3d_3 = \mu_1K^2H\left(1 - \frac{\mu_2}{\mu_0}\right)\left(1 - \frac{\rho C^2}{\mu_0}\right)$$

$$a_4d_4 = 0$$

$$b_1C_1 = n\left(1 - \frac{\mu_2}{\mu_0}\right)\sinh(-KH)$$

$$b_2C_2 = \left(1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2} - \frac{\varepsilon}{2}\right)\cosh(-KH)$$

$$b_3C_3 = \left(1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2}\right)\left(\frac{\varepsilon\mu_0}{2\mu_1} - 1\right)\left[1 + \frac{K^2H^2}{2}\left(1 - \frac{\rho C^2}{\mu_0}\right)\right]$$

$$b_4C_4 = \frac{KHn}{6}\left(1 - \frac{\mu_2}{\mu_0} - \frac{\varepsilon}{2}\right)\left(\frac{\varepsilon\mu_0}{2\mu_1} - 1\right)\left[-1 + K^2H^2\left(\frac{\rho C^2}{\mu_0} - 1\right)\right]$$

Case II: In the absence of substratum, i.e. if $H \to 0$, we have

$$a_1 = \frac{1}{K}\left[K + bm - K\left(\frac{\mu_2}{\mu_0} - \frac{\varepsilon\delta}{2}\right)\right]$$

$$a_2 = 1 - \frac{\mu_2}{\mu_0} - \frac{\varepsilon}{2}$$

$$a_3 = 1 - \frac{\mu_2}{\mu_0}$$

$$a_4 = \frac{1}{b}\left[K - bm - K\left(\frac{\mu_2}{\mu_0} - \frac{\varepsilon\delta}{2}\right) - \frac{\varepsilon K}{2}\right]$$

$$b_1 = \frac{1}{K}\left[Kn + bm - \frac{\mu_2}{\mu_0}Kn\right]$$

$$b_2 = 1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2} - \frac{\varepsilon}{2}$$

$$b_3 = 1 - \frac{\mu_2}{\mu_0} + \frac{\varepsilon\delta}{2}$$

$$b_4 = \frac{1}{b}\left(Kn + bm - \frac{\mu_2}{\mu_0}Kn - \frac{\varepsilon Kn}{2}\right)$$

$$C_1 = 0$$

$$C_2 = 1$$

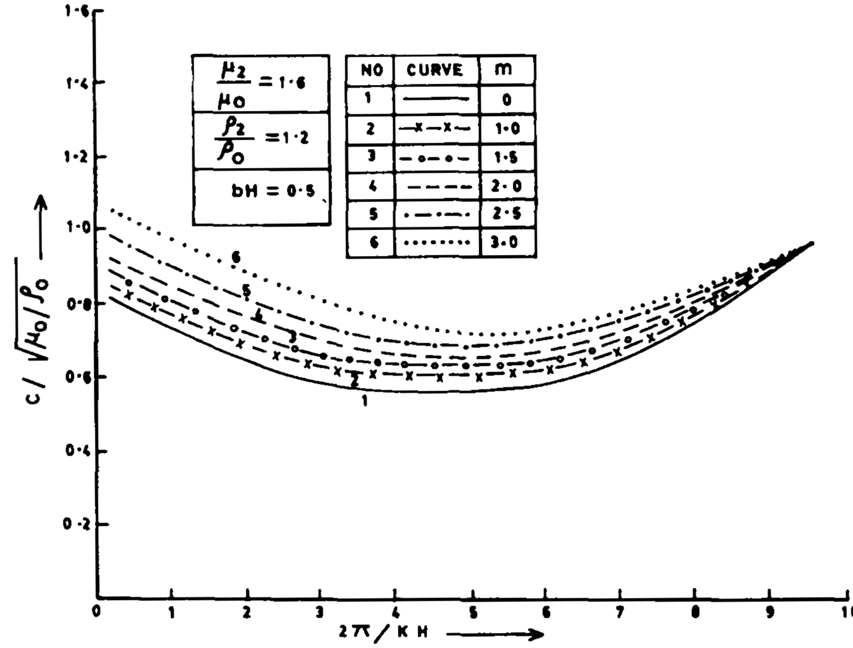

Figure 2. A comparison of the Rayleigh wave dispersion curve in a heterogeneous incompressible substratum over an incompressible homogeneous half-space

$$C_3 = \left(\frac{\varepsilon\mu_0}{2\mu_1} - 1\right)$$

$$C_4 = 0$$

$$d_1 = K\left(\mu_1 - \frac{\varepsilon\mu_0}{2}\right)$$

$$d_2 = -\mu_0 bm$$

$$d_3 = \mu_0 bm$$

$$d_4 = -\mu_1 b$$

and hence the velocity equation takes the form

$$\left(2 - \frac{\rho_2 C^2}{\mu_2}\right)^2 = 4\left(1 - \frac{\rho_2 C^2}{\mu_2}\right)^{1/2} \tag{37}$$

which is the well-known Rayleigh wave equation for an incompressible medium.

Examples of application and discussion of results

Using the parametric relations $\mu_2/\mu_0 = 1{\cdot}6$, $\rho_2/\rho_0 = 1{\cdot}2$ and $bH = 0{\cdot}5$ the values of μ_1/μ_0 have been calculated from the relation $\mu_1 = \mu_0(1 - bH)^m$ for different values of m starting from 0 to 3, and hence the velocities of Rayleigh waves have been calculated from equation (36) for different values of $2\pi/KH$. The results are presented in Figure 2. The curves show that an increase in the

non-homogeneity increases the velocity. These curves also suggest that as $2\pi/KH$ increases, i.e. the thickness of the superficial layer decreases, the velocity of Rayleigh waves tends to the value $0{\cdot}95(\mu_2/\rho_2)^{1/2}$ and coincides with the velocity of Rayleigh waves in a homogeneous half-space.

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